A BOUNDARY VALUE PROBLEM ON THE HALF-LINE FOR HIGHER ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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This paper is dedicated to the memory of Stephanos Mich. Philos, a great educator and scholar

ABSTRACT. This article is devoted to the study of the existence of solutions as well as the existence and uniqueness of solutions to a boundary value problem on the half-line for higher order nonlinear ordinary differential equations. By the use of the Schauder-Tikhonov theorem, an existence result is obtained; also, via the Banach contraction principle, an existence and uniqueness criterion is established. These two results are applied, in particular, to the specific class of higher order nonlinear ordinary differential equations of Emden-Fowler type and to the special case of higher order linear ordinary differential equations, respectively. Moreover, some (general or specific) examples demonstrating the applicability of our results are given.

1. INTRODUCTION AND PRELIMINARIES

In the asymptotic theory of (ordinary or, more generally, functional) differential equations, it is of great interest to study the problem of the existence of solutions with prescribed asymptotic behavior. This problem has been the subject of many investigations; we restrict ourselves to mentioning the papers [1, 2, 6, 10–13, 15–38, 41–45] (see also the references cited therein). This paper deals with the existence and the existence and uniqueness of solutions with a specific prescribed asymptotic behavior for higher order nonlinear ordinary differential equations. So, our work is a continuation of the study in the above mentioned papers.

In particular, one main center of interest has been in the problem of the existence of global solutions (i.e., solutions on the whole given interval) with prescribed asymptotic behavior. In the last few years, the existence of global solutions with prescribed asymptotic behavior is usually formulated as the existence of solutions to boundary value problems on the half-line; see, for example, the recent papers [6, 18–20, 29–32, 42–44] (and the references cited therein). The relation between the present article and the papers [19, 20, 29–32] is described below. As it concerns the papers [6, 18, 42–44], they are closely related to the articles [19, 20].

Recently, Mavridis, the author and Tsamatos [19] studied the existence of solutions to a boundary value problem on the half-line for second order nonlinear delay differential equations; the basic tool in the approach in [19] is the classical Schauder theorem. Later, in a different direction, the same authors [20] used the Krasnosel'skiĭ fixed point theorem to investigate the existence of multiple positive

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solutions to a boundary value problem on the half-line for second order nonlinear delay differential equations. By the use of the Schauder-Tikhonov theorem and the Banach contraction principle, the author [30] has recently studied the problem of the existence of solutions and of the existence and uniqueness of solutions to a boundary value problem on the half-line for nonlinear two-dimensional delay differential systems. The results obtained in [30] include, as special cases, those given in [19] for second order nonlinear delay differential equations.

Also, recently, the author [29] established sufficient conditions that guarantee the existence of positive increasing solutions to a boundary value problem on the half-line for second order nonlinear delay differential equations with positive delays. Motivated by the work in [29] (and, in a sense, by the author's paper [30]), the author [31] has considered a boundary value problem on the half-line for nonlinear two-dimensional delay differential systems with positive delays, and has obtained sufficient conditions for the existence of positive solutions. The results contained in [29] can be derived, as special consequences, from the ones established in [31]. Moreover, very recently, the author [32] extended the results given in [29] to the more general case of higher order nonlinear delay differential equations. More precisely, in [32], the author obtained sufficient conditions that guarantee the existence of positive solutions to a boundary value problem on the half-line for n-th order ($n > 2$) nonlinear delay differential equations with positive delays. Note that the assumption of the positivity of the delays is an essential condition to the approach in [29, 31, 32]; hence, the results obtained in these works are not applicable to the corresponding boundary value problems on the half-line for nonlinear ordinary differential equations or systems. We also notice that the approach in the papers [29, 31, 32] is elementary and is essentially based on an old idea which appeared in an author's paper in 1981.

The present paper is essentially motivated by the works in [19, 20, 29–32]. Here, a boundary value problem on the half-line for n-th order ($n > 1$) nonlinear ordinary differential equations is considered, and the problems of the existence of solutions as well as of the existence and uniqueness of solutions are treated. Our approach is based on the use of the Schauder-Tikhonov theorem (for the problem of the existence of solutions) and the Banach contraction principle (for the problem of the existence and uniqueness of solutions). Our results in this paper will be established in the classical case of ordinary differential equations. Following the lines of this work and using some elements of the techniques applied in [19, 30], one can extend the results of the present paper to the more general case of a corresponding boundary value problem on the half-line for n-th order ($n > 1$) nonlinear delay differential equations.

Consider the n-th order ($n > 1$) nonlinear ordinary differential equation

$$x^{(n)}(t) + f\left(t, x(t), x'(t), ..., x^{(n-1)}(t)\right) = 0, \tag{1.1}$$

where f *is a continuous real-valued function on* $[0, \infty) \times \mathbb{R}^n$. Our interest will be concentrated on global solutions of (1.1), i.e., on solutions of (1.1) on the *whole* interval $[0, \infty)$. Together with the differential equation (1.1), we specify the initial condition

$$x(0) = x'(0) = ... = x^{(n-2)}(0) = 0. \tag{1.2}$$

Moreover, along with (1.1), we impose a condition of the form

$$\lim_{t\to\infty} x^{(n-1)}(t) = \xi, \tag{1.3}$$

where ξ *is a given real number*. It must be noted that (1.3) implies that

$$\lim_{t\to\infty} \frac{x^{(k)}(t)}{t^{n-1-k}} = \frac{1}{(n-1-k)!}\xi \quad (k = 0, 1, ..., n-2).$$

The differential equation (1.1) together with the conditions (1.2) and (1.3) constitute a *boundary value problem* (*BVP*, for short) *on the half-line*. A solution on $[0, \infty)$ of (1.1) satisfying (1.2) and (1.3) is said to be a *solution* of the boundary value problem (1.1)–(1.3) or, more briefly, a *solution* of the BVP (1.1)–(1.3).

Here, we shall introduce a useful notation that will be used throughout the paper without mentioning it any further. For any continuous real-valued function h defined on the interval $[0, \infty)$, we set

$$I_k[h](t) = \int_0^t \frac{(t-s)^{n-2-k}}{(n-2-k)!} h(s)\,ds \quad \text{for } t \geq 0 \qquad (k = 0, 1, ..., n-2).$$

For our convenience, we consider the integrodifferential equation

$$y'(t) + f(t, I_0[y](t), I_1[y](t), ..., I_{n-2}[y](t), y(t)) = 0. \tag{1.4}$$

We are interested in solutions of (1.4) on the *whole* interval $[0, \infty)$. With the integrodifferential equation (1.4), we associate the condition

$$\lim_{t\to\infty} y(t) = \xi. \tag{1.5}$$

Equations (1.4) and (1.5) constitute a *boundary value problem* (*BVP*, for short) *on the half-line*. A solution on $[0, \infty)$ of the integrodifferential equation (1.4) that satisfies the condition (1.5) will be called a *solution* of the boundary value problem (1.4) and (1.5) or, more briefly, a *solution* of the BVP (1.4) and (1.5).

Furthermore, let us consider the integral equation

$$y(t) = \xi + \int_t^\infty f(u, I_0[y](u), I_1[y](u), ..., I_{n-2}[y](u), y(u))\,du. \tag{1.6}$$

Our interest is concentrated on solutions of (1.6) on the *whole* interval $[0, \infty)$.

The following lemma plays a crucial role in our approach in the present paper.

Lemma 1.1. *If x is a solution of the BVP (1.1)–(1.3), then the function y defined by*

$$y(t) = x^{(n-1)}(t) \quad \textit{for } t \geq 0 \tag{1.7}$$

is a solution of the BVP (1.4) and (1.5). Conversely, if y is a solution of the BVP (1.4) and (1.5), then the function x defined by

$$x(t) = \int_0^t \frac{(t-s)^{n-2}}{(n-2)!} y(s)\,ds \quad \textit{for } t \geq 0 \tag{1.8}$$

is a solution of the BVP (1.1)–(1.3). Furthermore, a continuous real-valued function y defined on the interval $[0, \infty)$ is a solution of the BVP (1.4) and (1.5) if and only if it is a solution on $[0, \infty)$ of the integral equation (1.6).

Proof. Let x be a solution of the BVP (1.1)−(1.3), and define the function y by (1.7). By taking into account the initial condition (1.2), after some elementary calculations, we can show that

$$x^{(k)}(t) = \int_0^t \frac{(t-s)^{n-2-k}}{(n-2-k)!} x^{(n-1)}(s)\, ds \quad \text{for } t \geq 0 \qquad (k = 0, 1, ..., n-2).$$

Thus, in view of (1.7), we have

$$x^{(k)}(t) = I_k[y](t) \quad \text{for } t \geq 0 \qquad (k = 0, 1, ..., n-2). \tag{1.9}$$

By (1.7) and (1.9), the fact that x is a solution on $[0, \infty)$ of the differential equation (1.1) implies that y is a solution on $[0, \infty)$ of the integrodifferential equation (1.4). Moreover, because of (1.7), the fact that x satisfies condition (1.3) means that y satisfies condition (1.5). So, the function y is a solution of the BVP (1.4) and (1.5).

Conversely, let us consider a solution y of the BVP (1.4) and (1.5), and define the function x by (1.8). From (1.8) it follows easily that x satisfies (1.9) and (1.7). Because of (1.7) and (1.9), the fact that y is a solution on $[0, \infty)$ of the integrodifferential equation (1.4) guarantees that x is a solution on $[0, \infty)$ of the differential equation (1.1). Also, it follows from (1.9) that x satisfies the initial condition (1.2). Moreover, by (1.7), the fact that y satisfies condition (1.5) ensures that x satisfies condition (1.3). Hence, x is a solution of the BVP (1.1)−(1.3).

Now, let y be a continuous real-valued function defined on the interval $[0, \infty)$. Assume, first, that y is a solution on $[0, \infty)$ of the integral equation (1.6). Then we immediately see that y satisfies (1.4) for all $t \geq 0$, i.e., that y is a solution on $[0, \infty)$ of the integrodifferential equation (1.4). Also, we see that y satisfies condition (1.5). Thus, y is a solution of the BVP (1.4) and (1.5). Next, let us assume that y is a solution of the BVP (1.4) and (1.5). Then, it follows from (1.4) that

$$\lim_{T \to \infty} y(T) - y(t) + \int_t^\infty f\left(u, I_0[y](u), I_1[y](u), ..., I_{n-2}[y](u), y(u)\right) du = 0$$

for every $t \geq 0$. So, by taking into account condition (1.5), we see that y satisfies (1.6) for all $t \geq 0$, i.e., that y is a solution on $[0, \infty)$ of the integral equation (1.6).

The proof of our lemma is complete.

Our main results in this paper are formulated as two theorems (Theorems 2.1 and 2.2), which will be stated in Section 2. Theorem 2.1 establishes sufficient conditions for the BVP (1.1)−(1.3) to have at least one solution, while Theorem 2.2 provides conditions that guarantee the existence of a unique solution of the BVP (1.1)−(1.3). Section 2 contains also some useful comments about these two theorems. The proofs of Theorems 2.1 and 2.2 will be presented in Section 3. Section 4 is devoted to the application of Theorem 2.1 to the specific class of n-th order $(n > 1)$ nonlinear ordinary differential equations of Emden-Fowler type as well as to the application of Theorem 2.2 to the special case of n-th order $(n > 1)$ linear ordinary differential equations. Moreover, Section 4 includes some (general or specific) examples, which demonstrate the applicability of our theorems.

2. STATEMENT OF THE MAIN RESULTS AND COMMENTS

Our main results are the following two theorems.

Theorem 2.1. *Assume that*

$$|f(t, z_0, z_1, ..., z_{n-1})| \le F(t, |z_0|, |z_1|, ..., |z_{n-1}|) \quad \text{for all } (t, z_0, z_1, ..., z_{n-1}) \in [0, \infty) \times \mathbb{R}^n, \tag{2.1}$$

where F is a nonnegative continuous real-valued function on $[0, \infty) \times [0, \infty)^n$. Also, assume that, for each $t \ge 0$, the function $F(t, \cdot, \cdot, ..., \cdot)$ is increasing on $[0, \infty)^n$ in the sense that $F(t, z_0, z_1, ..., z_{n-1}) \le F(t, w_0, w_1, ..., w_{n-1})$ for any $(z_0, z_1, ..., z_{n-1})$, $(w_0, w_1, ..., w_{n-1})$ in $[0, \infty)^n$ with $z_0 \le w_0$, $z_1 \le w_1$, ..., $z_{n-1} \le w_{n-1}$.

Let there exist a real number c with $c > |\xi|$ so that

$$\int_0^\infty F\left(t, \frac{c}{(n-1)!}t^{n-1}, \frac{c}{(n-2)!}t^{n-2}, ..., ct, c\right) dt \le c - |\xi|. \tag{2.2}$$

Then the BVP (1.1)–(1.3) has at least one solution x such that

$$\frac{-c + |\xi| + \xi}{(n-1-i)!} t^{n-1-i} \le x^{(i)}(t) \le \frac{c - |\xi| + \xi}{(n-1-i)!} t^{n-1-i} \quad \text{for every } t \ge 0 \quad (i = 0, 1, ..., n-1). \tag{2.3}$$

Theorem 2.2. *Let the assumptions of Theorem 2.1 hold. Furthermore, let the following generalized Lipschitz condition be satisfied:*

$$\begin{aligned} &|f(t, z_0, z_1, ..., z_{n-1}) - f(t, w_0, w_1, ..., w_{n-1})| \\ &\quad \le L(t) \max\{|z_0 - w_0|, |z_1 - w_1|, ..., |z_{n-1} - w_{n-1}|\} \\ &\quad \text{for all } (t, z_0, z_1, ..., z_{n-1}), \ (t, w_0, w_1, ..., w_{n-1}) \text{ in } [0, \infty) \times \mathbb{R}^n, \end{aligned} \tag{2.4}$$

where L is a nonnegative continuous real-valued function on the interval $[0, \infty)$. Assume that there exists a positive continuous real-valued function p on the interval $[0, \infty)$ with

$$0 < \liminf_{t \to \infty} p(t) \le \limsup_{t \to \infty} p(t) < \infty \tag{2.5}$$

such that

$$\sup_{t \ge 0} \left[\frac{1}{p(t)} \int_t^\infty L(u) \max\{I_0[p](u), I_1[p](u), ..., I_{n-2}[p](u), p(u)\} du\right] < 1. \tag{2.6}$$

Let there exist a real number c with $c > |\xi|$ so that (2.2) holds. Then the BVP (1.1)–(1.3) has exactly one solution x with

$$\left|x^{(n-1)}(t)\right| \le c \quad \text{for all } t \ge 0; \tag{2.7}$$

this unique solution x is such that (2.3) holds.

Remark 2.3. In the conclusions of Theorems 2.1 and 2.2, the solution x of the BVP (1.1)–(1.3) satisfies (2.3), i.e., it is such that

$$\text{(2.8)} \quad \frac{-c+|\xi|+\xi}{(n-1-k)!}t^{n-1-k} \leq x^{(k)}(t) \leq \frac{c-|\xi|+\xi}{(n-1-k)!}t^{n-1-k} \quad \text{for every } t \geq 0 \quad (k=0,1,...,n-2)$$

and

$$\text{(2.9)} \quad -c+|\xi|+\xi \leq x^{(n-1)}(t) \leq c-|\xi|+\xi \quad \text{for every } t \geq 0.$$

It is remarkable that, because of the initial condition (1.2), *inequalities* (2.8) *are consequences of inequalities* (2.9).

Remark 2.4. We will present here some important observations about the solution x of the BVP (1.1)–(1.3) in the conclusions of Theorems 2.1 and 2.2; this solution satisfies (2.3).

Assume that $\xi > 0$. Then (2.3) is written as

$$\text{(2.3}'\text{)} \quad \frac{-c+2\xi}{(n-1-i)!}t^{n-1-i} \leq x^{(i)}(t) \leq \frac{c}{(n-1-i)!}t^{n-1-i} \quad \text{for every } t \geq 0 \quad (i=0,1,...,n-1).$$

Furthermore, *in addition to the hypothesis* $c > \xi$, *let us suppose that* $c < 2\xi$. *We thus have* $0 < \xi < c < 2\xi$. Then, (2.3′) guarantees that $x^{(k)}$ $(k = 0, 1, ..., n-2)$ *are positive on the interval* $(0, \infty)$ *and such that* $\lim_{t\to\infty} x^{(k)}(t) = \infty$ $(k = 0, 1, ..., n-2)$. Also, from (2.3′) it follows that $x^{(n-1)}$ *is positive on the interval* $[0, \infty)$.

Analogously, *in the case where* $2\xi < -c < \xi < 0$, we can see that $x^{(k)}$ $(k = 0, 1, ..., n-2)$ *are negative on* $(0, \infty)$ *and such that* $\lim_{t\to\infty} x^{(k)}(t) = -\infty$ $(k = 0, 1, ..., n-2)$, and that $x^{(n-1)}$ *is negative on* $[0, \infty)$.

Remark 2.5. Theorem 2.2 can be applied with $p(t) = 1$ for $t \geq 0$. In this special case, (2.5) is automatically fulfilled and assumption (2.6) becomes

$$\text{(2.10)} \quad \int_0^\infty L(t) \max\left\{\frac{t^{n-1}}{(n-1)!}, \frac{t^{n-2}}{(n-2)!}, ..., t, 1\right\} < 1.$$

More precisely, we have the following particular result:

Let the assumptions of Theorem 2.1 *hold. Furthermore, let the generalized Lipschitz condition* (2.4) *be satisfied, where* L *is a nonnegative continuous real-valued function on the interval* $[0, \infty)$ *such that* (2.10) *holds. Let there exist a real number* c *with* $c > |\xi|$ *so that* (2.2) *holds. Then the BVP* (1.1)–(1.3) *has exactly one solution* x *satisfying* (2.7); *this unique solution* x *is such that* (2.3) *holds.*

Observe that condition (2.10) implies that

$$\text{(2.11)} \quad \int_0^\infty t^{n-1} L(t) < \infty.$$

It is an open problem whether the assertion of the above particular result remains valid with the weaker assumption (2.11) in place of (2.10). To investigate this open problem, one must apply Theorem 2.2 with a suitable positive continuous real-valued function p on the interval $[0, \infty)$ that satisfies (2.5). In other words, one must use a renormalization procedure due to Bielecki [4]; this technique is very useful in obtaining global existence criteria and in studying several stability problems (see, for example, the books by Corduneanu [8, 9], and the recent papers by Ehrnström [10–12] and Wahlén [41]).

3. PROOFS OF THE THEOREMS

To prove Theorem 2.1 we will use the fixed point technique, by applying the Schauder-Tikhonov theorem. This fixed point theorem is essentially the extension to function spaces of Brouwer's well-known fixed point theorem for mappings in Euclidean spaces. This extension, first made by Birkhoff and Kellogg [5] and then in greater generality by Schauder [39] and Tychonoff [40], will be stated here only for the special case which we require.

Let V be a set of real-valued functions defined on an interval J, and let t_0 be a point of J. The set V is said to be *equicontinuous at* t_0 if, for each $\epsilon > 0$, there exists a corresponding $\delta \equiv \delta(\epsilon) > 0$ such that, for all functions v in V, it holds that $|v(t) - v(t_0)| < \epsilon$ for every $t \in J$ with $|t - t_0| < \delta$. Also, the set V is called *bounded at* t_0 if there exists a real number $\Theta > 0$ such that $|v(t_0)| \leq \Theta$ for all functions $v \in V$.

The Schauder-Tikhonov theorem. *Let μ be a fixed positive continuous real-valued function on an interval J, and let Y be the set of all continuous real-valued functions y defined on J which satisfy*

$$|y(t)| \leq \mu(t) \quad \textit{for all } t \in J.$$

Let M be a mapping of Y into intself with the properties:

(i) *the mapping M is continuous in the sense that, for each function y in Y and any sequence $(y_m)_{m\geq 1}$ of functions in Y, we have: if $\lim_{m\to\infty} y_m = y$ uniformly on every compact subinterval of J, then $\lim_{m\to\infty} My_m = My$ uniformly on every compact subinterval of J;*

(ii) *the image set MY is equicontinuous at every point of J.*

Then the mapping M has at least one fixed point in Y, i.e., there exists at least one y in Y with $y = My$.

The Schauder-Tikhonov theorem has been stated, in the form presented above, by Coppel [7; p. 9], under the additional assumption that *the image set MY is bounded at every point of J*. But this assumption is not needed. Indeed, as $MY \subseteq Y$, from the definition of Y it follows that $|(My)(t)| \leq \mu(t)$ for all functions y in Y and every $t \in J$, which implies that MY is always bounded at every point of J. A proof of the Schauder-Tikhonov theorem stated above, which is based on the use of Brouwer's theorem for mappings in Euclidean spaces, can be found in [7; pp. 9–10].

The proof of Theorem 2.2 is based on the use of the well-known Banach contraction principle (see Banach [3]; see also Kartsatos [14; p. 27]).

The Banach contraction principle. *Let E be a Banach space and Y any nonempty closed subset of E. If M is a contraction of Y into itself, then the mapping M has exactly one fixed point in Y, i.e., there exists a unique y in Y such that $y = MY$.*

In order to prove Theorems 2.1 and 2.2, we will first establish the following proposition.

Proposition 3.1. *Let the assumptions of Theorem 2.1 hold.*
Let c be a positive real number such that

$$\int_0^\infty F\left(t, \frac{c}{(n-1)!}t^{n-1}, \frac{c}{(n-2)!}t^{n-2}, ..., ct, c\right) dt < \infty. \tag{3.1}$$

Also, let Y be the set of all continuous real-valued functions y defined on the interval $[0, \infty)$ which satisfy

$$|y(t)| \leq c \quad \text{for all } t \geq 0. \tag{3.2}$$

Then the formula
(3.3)

$$(My)(t) = \xi + \int_t^\infty f(u, I_0[y](u), I_1[y](u), ..., I_{n-2}[y](u), y(u))\, du \quad \text{for } t \geq 0$$

makes sense for any function y in Y, and this formula defines a mapping M of Y into the set of all continuous real-valued functions defined on $[0, \infty)$. Moreover, M has the properties:

(i) *the mapping M is continuous in the sense of the requirement* (i) *of the Schauder-Tikhonov theorem;*

(ii) *the image set MY is equicontinuous and bounded at every point of $[0, \infty)$.*

Proof. Consider an arbitrary function y in Y. By the definition of Y, the function y satisfies (3.2). By using (3.2), for any $k \in \{0, 1, ..., n-2\}$ and every $t \geq 0$, we obtain

$$\begin{aligned} |I_k[y](t)| &= \left| \int_0^t \frac{(t-s)^{n-2-k}}{(n-2-k)!} y(s)\, ds \right| \leq \int_0^t \frac{(t-s)^{n-2-k}}{(n-2-k)!} |y(s)|\, ds \\ &\leq c \int_0^t \frac{(t-s)^{n-2-k}}{(n-2-k)!} ds = \frac{c}{(n-1-k)!} t^{n-1-k}. \end{aligned}$$

Because of these inequalities and (3.2) and the assumption that, for each $t \geq 0$, the function $F(t, \cdot, \cdot, ..., \cdot)$ is increasing on $[0, \infty)^n$, we get

$$\begin{aligned} &F(t, |I_0[y](t)|, |I_1[y](t)|, ..., |I_{n-2}[y](t)|, |y(t)|) \\ &\leq F\left(t, \frac{c}{(n-1)!}t^{n-1}, \frac{c}{(n-2)!}t^{n-2}, ..., ct, c\right) \quad \text{for all } t \geq 0. \end{aligned}$$

Together with assumption (2.1) this guarantees that

$$\begin{aligned} &|f(t, I_0[y](t), I_1[y](t), ..., I_{n-2}[y](t), y(t))| \\ &\leq F\left(t, \frac{c}{(n-1)!}t^{n-1}, \frac{c}{(n-2)!}t^{n-2}, ..., ct, c\right) \quad \text{for every } t \geq 0. \end{aligned} \tag{3.4}$$

In view of (3.4) and hypothesis (3.1), we always have

$$\int_0^\infty |f(t, I_0[y](t), I_1[y](t), ..., I_{n-2}[y](t), y(t))|\, dt < \infty.$$

Hence, we immediately see that the formula (3.3) makes sense for any function y in Y, and this formula defines a mapping M of Y into the set of all continuous real-valued functions defined on the interval $[0, \infty)$.

Now, by using (3.3) and (3.4), we have, for any function y in Y and every $t_0 \geq 0$ and $t \geq 0$,

$$\begin{aligned}
&\left|(My)(t) - (My)(t_0)\right| \\
=\ & \left| \left[\xi + \int_t^\infty f\left(u, I_0[y](u), I_1[y](u), ..., I_{n-2}[y](u), y(u)\right) du \right] \right. \\
&\left. - \left[\xi + \int_{t_0}^\infty f\left(u, I_0[y](u), I_1[y](u), ..., I_{n-2}[y](u), y(u)\right) du \right] \right| \\
=\ & \left| \int_{t_0}^t f\left(u, I_0[y](u), I_1[y](u), ..., I_{n-2}[y](u), y(u)\right) du \right| \\
\leq\ & \left| \int_{t_0}^t \left| f\left(u, I_0[y](u), I_1[y](u), ..., I_{n-2}[y](u), y(u)\right) \right| du \right| \\
\leq\ & \left| \int_{t_0}^t F\left(u, \frac{c}{(n-1)!}u^{n-1}, \frac{c}{(n-2)!}u^{n-2}, ..., cu, c\right) du \right|.
\end{aligned}$$

So, by taking into account hypothesis (3.1), we can easily conclude that the image set MY is equicontinuous at every point $t_0 \geq 0$. Furthermore, by using again (3.3) and (3.4), for any function y in Y and every $t_0 \geq 0$, we obtain

$$\begin{aligned}
\left|(My)(t_0)\right| &= \left| \xi + \int_{t_0}^\infty f\left(u, I_0[y](u), I_1[y](u), ..., I_{n-2}[y](u), y(u)\right) du \right| \\
&\leq |\xi| + \int_{t_0}^\infty \left| f\left(u, I_0[y](u), I_1[y](u), ..., I_{n-2}[y](u), y(u)\right) \right| du \\
&\leq |\xi| + \int_{t_0}^\infty F\left(u, \frac{c}{(n-1)!}u^{n-1}, \frac{c}{(n-2)!}u^{n-2}, ..., cu, c\right) du.
\end{aligned}$$

Thus, by hypothesis (3.1), MY is also bounded at every point $t_0 \geq 0$. We have thus seen that the mapping M has the property (ii).

Next, we shall prove that M has the property (i). Let y be an arbitrary function in Y and $(y_m)_{m\geq 1}$ be any sequence of functions in Y with $\lim_{m\to\infty} y_m = y$ uniformly on every compact subinterval of $[0, \infty)$. By (3.4), we have

$$\begin{aligned}
&\left| f\left(t, I_0[y_m](t), I_1[y_m](t), ..., I_{n-2}[y_m](t), y_m(t)\right) \right| \\
&\leq F\left(t, \frac{c}{(n-1)!}t^{n-1}, \frac{c}{(n-2)!}t^{n-2}, ..., ct, c\right) \quad \text{for every } t \geq 0 \qquad (m = 1, 2, ...).
\end{aligned}$$

So, because of hypothesis (3.1), we can apply the Lebesgue dominated convergence theorem to obtain, for every $t \geq 0$,

$$\begin{aligned}
&\lim_{m\to\infty} \int_t^\infty f\left(u, I_0[y_m](u), I_1[y_m](u), ..., I_{n-2}[y_m](u), y_m(u)\right) du \\
&\qquad = \int_t^\infty f\left(u, I_0[y](u), I_1[y](u), ..., I_{n-2}[y](u), y(u)\right) du.
\end{aligned}$$

Hence, by taking into account the definition of the mapping M by (3.3), we get $\lim_{m\to\infty} My_m = My$ pointwise on $[0, \infty)$. It remains to show that $\lim_{m\to\infty} My_m = My$ uniformly on every compact subinterval of $[0, \infty)$. To this end, let us consider an arbitrary subsequence $(My_{\lambda_m})_{m\geq 1}$ of $(My_m)_{m\geq 1}$. We have previously seen that MY is equicontinuous and bounded at every point of $[0, \infty)$. Thus, by the

Arzelà-Ascoli theorem, there exists a subsequence $(My_{\lambda_{\nu_m}})_{m\geq 1}$ of $(My_{\lambda_m})_{m\geq 1}$ and a continuous real-valued function v defined on the interval $[0,\infty)$ such that $\lim_{m\to\infty} My_{\lambda_{\nu_m}} = v$ uniformly on every compact subinterval of $[0,\infty)$. Since the uniform convergence on every compact subinterval of $[0,\infty)$ implies the pointwise convergence on $[0,\infty)$ to the same limit function, we always have $v = My$. So, $(My_m)_{m\geq 1}$ converges to My uniformly on every compact subinterval of $[0,\infty)$. Consequently, the mapping M is continuous.

The proof of the proposition has been finished.

Now, we are in a position to present the proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. First of all, we observe that hypothesis (2.2) implies (3.1). Let Y be the set defined as in Proposition 3.1. By this proposition, the formula (3.3) makes sense for any function y in Y, and this formula defines a mapping M of Y into the set of all continuous real-valued functions defined on $[0,\infty)$. Moreover, M has the properties (i) and (ii) described in Proposition 3.1. We shall show that M is a mapping of Y into itself, i.e., that $MY \subseteq Y$. To this end, let us consider an arbitrary function y in Y. Then (3.4) holds true, and consequently form (3.3) we obtain, for $t \geq 0$,

$$\begin{aligned}
|(My)(t) - \xi| &= \left| \int_t^\infty f\left(u, I_0[y](u), I_1[y](u), ..., I_{n-2}[y](u), y(u)\right) du \right| \\
&\leq \int_t^\infty \left| f\left(u, I_0[y](u), I_1[y](u), ..., I_{n-2}[y](u), y(u)\right) \right| du \\
&\leq \int_0^\infty \left| f\left(u, I_0[y](u), I_1[y](u), ..., I_{n-2}[y](u), y(u)\right) \right| du \\
&\leq \int_0^\infty F\left(u, \frac{c}{(n-1)!}u^{n-1}, \frac{c}{(n-2)!}u^{n-2}, ..., cu, c\right) du.
\end{aligned}$$

Hence, in view of hypothesis (2.2), we have

$$|(My)(t) - \xi| \leq c - |\xi| \quad \text{for every } t \geq 0. \tag{3.5}$$

The last inequality implies that $|(My)(t)| \leq c$ for all $t \geq 0$ and so My belongs to Y. We have thus proved that, for any y in Y, $My \in Y$, i.e., that $MY \subseteq Y$.

Now we apply the Schauder-Tikhonov theorem to conclude that there exists at least one y in Y with $y = My$. By Lemma 1.1, y is a solution of the BVP (1.4) and (1.5), and the function x defined by (1.8) is a solution of the BVP (1.1)$-$(1.3). From (1.8) it follows easily that x satisfies (1.7). As $y \in Y$ and $y = My$, by taking into account (1.7), from (3.5) we obtain

$$\left| x^{(n-1)}(t) - \xi \right| \leq c - |\xi| \quad \text{for every } t \geq 0.$$

Clearly, the last inequality coincides with (2.9). Finally, by using the initial condition (1.2), it is not difficult to verify that (2.9) implies (2.8). Hence, the solution x of the BVP (1.1)$-$(1.3) is such that (2.8) and (2.9) hold, i.e., x satisfies (2.3).

The proof of the theorem is complete.

Proof of Theorem 2.2. Consider the Banach space $E \equiv BC([0,\infty), \mathbb{R})$ of all bounded continuous real-valued functions on the interval $[0,\infty)$, endowed with the usual sup-norm $\|\cdot\|$ defined by

$$\|v\| = \sup_{t\geq 0} |v(t)| \quad \text{for } v \in BC([0,\infty),\mathbb{R}).$$

Consider also the set Y defined as in Proposition 3.1. We immediately see that $Y = \{y \in BC([0,\infty),\mathbb{R}) : \|y\| \leq c\}$. The set Y is a nonempty closed subset of $BC([0,\infty),\mathbb{R})$.

As hypothesis (2.2) implies (3.1), Proposition 3.1 guarantees that the formula (3.3) makes sense for any function y in Y, and this formula defines a mapping M of Y into the set of all continuous real-valued functions defined on $[0,\infty)$, As in the proof of Theorem 2.1, we can use (2.2) to show that M is a mapping of Y into itself.

From hypothesis (2.5) it follows that there exist real constants α and β with $0 < \alpha \leq \beta$ such that

$$\alpha \leq p(t) \leq \beta \quad \text{for every } t \geq 0.$$

Thus, for any function v in $BC([0,\infty),\mathbb{R})$, we have

$$\frac{1}{\beta}|v(t)| \leq \frac{|v(t)|}{p(t)} \leq \frac{1}{\alpha}|v(t)| \quad \text{for all } t \geq 0$$

and consequently

$$\frac{1}{\beta}\|v\| \leq \sup_{t\geq 0}\frac{|v(t)|}{p(t)} \leq \frac{1}{\alpha}\|v\|.$$

Hence, the formula

$$\|v\|_p = \sup_{t\geq 0}\frac{|v(t)|}{p(t)} \quad \text{for } v \in BC([0,\infty),\mathbb{R})$$

defines a norm $\|\cdot\|_p$ in $BC([0,\infty),\mathbb{R})$ that is equivalent to the sup-norm $\|\cdot\|$. So, $E \equiv BC([0,\infty),\mathbb{R})$ is a Banach space with respect to the norm $\|\cdot\|_p$, and Y is a nonempty closed subset of $BC([0,\infty),\mathbb{R})$ with respect to the norm $\|\cdot\|_p$.

Now, we shall prove that the mapping M is a contraction with respect to the norm $\|\cdot\|_p$. For this purpose, let us consider two arbitrary functions y and $\widetilde{y}$ in Y. By using assumption (2.4), from the definition of the mapping M by (3.3), we obtain, for every $t \geq 0$,

$$\begin{aligned}
&\frac{|(My)(t) - (M\widetilde{y})(t)|}{p(t)} \\
&= \frac{1}{p(t)}\left|\left[\xi + \int_t^\infty f(u, I_0[y](u), I_1[y](u), ..., I_{n-2}[y](u), y(u))\,du\right]\right. \\
&\quad \left. - \left[\xi + \int_t^\infty f(u, I_0[\widetilde{y}](u), I_1[\widetilde{y}](u), ..., I_{n-2}[\widetilde{y}](u), \widetilde{y}(u))\,du\right]\right| \\
&\leq \frac{1}{p(t)}\int_t^\infty |f(u, I_0[y](u), I_1[y](u), ..., I_{n-2}[y](u), y(u)) \\
&\quad - f(u, I_0[\widetilde{y}](u), I_1[\widetilde{y}](u), ..., I_{n-2}[\widetilde{y}](u), \widetilde{y}(u))|\,du \\
&\leq \frac{1}{p(t)}\int_t^\infty L(u)\max\{|I_0[y](u) - I_0[\widetilde{y}](u)|, |I_1[y](u) - I_1[\widetilde{y}](u)|, ... \\
&\quad ..., |I_{n-2}[y](u) - I_{n-2}[\widetilde{y}](u)|, |y(u) - \widetilde{y}(u)|\}du
\end{aligned}$$

$$\leq \frac{1}{p(t)} \int_t^\infty L(u) \max \{I_0 [|y - \widetilde{y}|] (u), I_1 [|y - \widetilde{y}|] (u), ...$$
$$..., I_{n-2} [|y - \widetilde{y}|] (u), |y (u) - \widetilde{y} (u)|\} du$$
$$= \frac{1}{p(t)} \int_t^\infty L(u) \max \left\{ \int_0^u \frac{(u-s)^{n-2}}{(n-2)!} p(s) \frac{|y(s) - \widetilde{y}(s)|}{p(s)} ds, \right.$$
$$\int_0^u \frac{(u-s)^{n-3}}{(n-3)!} p(s) \frac{|y(s) - \widetilde{y}(s)|}{p(s)} ds, ...$$
$$\left. ..., \int_0^u p(s) \frac{|y(s) - \widetilde{y}(s)|}{p(s)} ds, p(u) \frac{|y(u) - \widetilde{y}(u)|}{p(u)} \right\} du$$
$$\leq \left[\frac{1}{p(t)} \int_t^\infty L(u) \max \{I_0 [p] (u), I_1 [p] (u), ..., I_{n-2} [p] (u), p(u)\} du \right] \|y - \widetilde{y}\|_p .$$

Thus, by hypothesis (2.6), there exists θ with $0 \leq \theta < 1$ such that

$$\|My - M\widetilde{y}\|_p \leq \theta \|y - \widetilde{y}\|_p .$$

This inequality holds true for all functions y and $\widetilde{y}$ in Y. Hence, M is a contraction with respect to the norm $\|\cdot\|_p$.

Let y be a function in Y with $y = My$. From the definition of the mapping M by (3.3) it follows that y is a solution on $[0, \infty)$ of the integral equation (1.6). Thus, Lemma 1.1 ensures that the function x defined by (1.8) is a solution of the BVP (1.1)−(1.3). It is easy to see that (1.8) implies that (1.7) is also valid. On the other hand, the definition of the set Y guarantees that (3.2) holds true. A combination of (1.7) and (3.2) leads to the fact that the solution x of the BVP (1.1)−(1.3) satisfies (2.7). Conversely, let x be a solution of the BVP (1.1)−(1.3) that satisfies (2.7). By Lemma 1.1, the function y defined by (1.7) is a solution on $[0, \infty)$ of the integral equation (1.6). Moreover, by (1.7), inequality (2.7) leads to (3.2). Therefore, y is a function in Y with $y = My$. After the above observations, the proof of our theorem can be accomplished by an application of the Banach contraction principle. Hence, there exists exactly one y in Y such that $y = My$. By the above analysis, the function x defined by (1.8) is the unique solution of the BVP (1.1)−(1.3) satisfying (2.7). As in the proof of Theorem 2.1, we conclude that this (unique) solution x of the BVP (1.1)−(1.3) is such that (2.3) holds.

The proof of the theorem is now complete.

4. APPLICATIONS AND EXAMPLES

Consider the n-th order ($n > 1$) nonlinear ordinary differential equation of Emden-Fowler type

$$x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t) \left|x^{(i)}(t)\right|^{\gamma_i} \operatorname{sgn} x^{(i)}(t) = 0 \tag{4.1}$$

as well as the n-th order ($n > 1$) linear ordinary differential equation

$$x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t) x^{(i)}(t) = 0, \tag{4.2}$$

where a_i $(i = 0, 1, ..., n-1)$ *are continuous real-valued functions on the interval* $[0, \infty)$, *and* γ_i $(i = 0, 1, ..., n-1)$ *are positive real numbers.*

We immediately see that the differential equation (4.1) as well as the differential equation (4.2) have the *zero solution* $x(t) = 0$ for $t \geq 0$. This solution satisfies the initial condition (1.2); moreover, when $\xi = 0$, it also satisfies the condition (1.3).

By specifying Theorem 2.1 to the particular case of the BVP (4.1), (1.2) and (1.3), we get the following corollary.

Corollary 4.1. *Let there exist a real number* c *with* $c > |\xi|$ *so that*

$$\sum_{i=0}^{n-1} \left[\frac{c}{(n-1-i)!} \right]^{\gamma_i} \int_0^{\infty} t^{(n-1-i)\gamma_i} |a_i(t)| \, dt \leq c - |\xi| . \tag{4.3}$$

Then the BVP (4.1), (1.2) *and* (1.3) *has at least one solution* x *such that* (2.3) *holds.*

Next, we shall apply Theorem 2.2 to the special case of the BVP (4.2), (1.2) and (1.3). In this case, the generalized Lipschitz condition (2.4) is satisfied with

$$L(t) = \sum_{i=0}^{n-1} |a_i(t)| \quad \text{for } t \geq 0.$$

Furthermore, we observe that the linear differential equation (4.2) can be obtained (as a special case) from (4.1) by taking $\gamma_i = 1$ for $i = 0, 1, ..., n-1$. In the special case of the BVP (4.2), (1.2) and (1.3), condition (4.3) becomes

$$c \sum_{i=0}^{n-1} \frac{1}{(n-1-i)!} \int_0^{\infty} t^{n-1-i} |a_i(t)| \, dt \leq c - |\xi| ,$$

i.e.,

$$c \left[1 - \sum_{i=0}^{n-1} \frac{1}{(n-1-i)!} \int_0^{\infty} t^{n-1-i} |a_i(t)| \, dt \right] \geq |\xi| . \tag{4.4}$$

Let the following condition be satisfied:

$$\sum_{i=0}^{n-1} \frac{1}{(n-1-i)!} \int_0^{\infty} t^{n-1-i} |a_i(t)| \, dt < 1. \tag{4.5}$$

Suppose, first, that $\xi = 0$. Then, by (4.5), we immediately see that (4.4) holds true for any $c > 0 = |\xi|$. Next, suppose that $\xi \neq 0$. Assume that at least one of the functions a_i $(i = 0, 1, ..., n-1)$ is not identically zero on $[0, \infty)$. This assumption guarantees that

$$\sum_{i=0}^{n-1} \frac{1}{(n-1-i)!} \int_0^{\infty} t^{n-1-i} |a_i(t)| \, dt > 0. \tag{4.6}$$

By (4.5) and (4.6), the formula

$$c = \frac{|\xi|}{1 - \sum_{i=0}^{n-1} \frac{1}{(n-1-i)!} \int_0^{\infty} t^{n-1-i} |a_i(t)| \, dt} \tag{4.7}$$

defines a real number c with $c > |\xi|$. For this real number c, inequality (4.4) holds true (as an equality). After the above observations, we are led to the next corollary.

Corollary 4.2. *Assume that there exists a positive continuous real-valued function p on the interval $[0,\infty)$ satisfying* (2.5) *and such that*

(4.8)

$$\sup_{t\geq 0}\left[\frac{1}{p(t)}\sum_{i=0}^{n-1}\int_t^{\infty}|a_i(u)|\max\{I_0[p](u), I_1[p](u), ..., I_{n-2}[p](u), p(u)\}\,du\right] < 1.$$

Moreover, assume that (4.5) *holds. Then we have:*

(i) *Let c be any positive real number. Then the BVP* (4.2), (1.2) *and*

$$\lim_{t\to\infty} x^{(n-1)}(t) = 0 \tag{4.9}$$

has exactly one solution x satisfying (2.7); *this unique solution is necessarily the zero solution $x(t)=0$ for $t\geq 0$.*

(ii) *Suppose that at least one of the functions a_i $(i=0,1,...,n-1)$ is not identically zero on $[0,\infty)$. Let $\xi\neq 0$, and let $c>|\xi|$ be the real number defined by* (4.7). *Then the BVP* (4.2), (1.2) *and* (1.3) *has exactly one solution x satisfying* (2.7); *this unique solution x is such that* (2.3) *holds.*

We notice here that the BVP (4.2), (1.2) and (4.9) admits the zero solution $x(t)=0$ for $t\geq 0$.

Remark 4.3. Choose $p(t)=1$ for $t\geq 0$ in Corollary 4.2. Then (2.5) is automatically fulfilled and assumption (4.8) becomes (cf. Remark 2.5)

$$\sum_{i=0}^{n-1}\int_0^{\infty}|a_i(t)|\max\left\{\frac{t^{n-1}}{(n-1)!}, \frac{t^{n-2}}{(n-2)!}, ..., t, 1\right\}dt < 1. \tag{4.10}$$

It is easy to see that (4.10) implies condition (4.5). So, we immediately obtain the following particular result:

Assume that (4.10) *holds. Then we have* (i) *and* (ii) *of Corollary* 4.2.

Now, in order to give two general examples (each one of which includes one specific example) demonstrating the applicability of Theorem 2.1 (more specifically, of Corollary 4.1), we will concentrate on the particular case of the n-th order $(n>1)$ nonlinear ordinary differential equation of Emden-Fowler type

$$x^{(n)}(t) + a(t)|x(t)|^{\gamma}\operatorname{sgn}x(t) = 0, \tag{4.11}$$

where *a is a continuous real-valued function on the interval $[0,\infty)$, and γ is a positive real number.* As it concerns the particular case of the BVP (4.11), (1.2) and (1.3), Corollary 4.1 is formulated as follows.

Let there exist a real number c with $c>|\xi|$ so that

$$\left[\frac{c}{(n-1)!}\right]^{\gamma}\int_0^{\infty} t^{(n-1)\gamma}|a(t)|\,dt \leq c-|\xi|. \tag{4.12}$$

Then the BVP (4.11), (1.2) *and* (1.3) *has at least one solution x such that* (2.3) *holds.*

In Examples 4.4 and 4.5 below, without mentioning it any further, *it will be assumed that the coefficient a is not identically zero on the interval $[0,\infty)$, and that $\xi\neq 0$.*

Examples 4.4 and 4.5 concern the Emden-Fowler type nonlinear ordinary differential equation (4.11) with $\gamma = \frac{1}{2}$ (sublinear case) and $\gamma = 2$ (superlinear case), respectively. The motivation for presenting these two examples is the fact that differential equations of Emden-Fowler type appear very often in applications. For example, Emden-Fowler equations arise in the study of gas dynamics and fluid mechanics. Also, such equations appear in the study of relativistic mechanics, nuclear physics and in the investigation of chemically reacting systems. The study of Emden-Fowler equations originates from earlier theories concerning gaseous dynamics in astrophysics around the turn of the century.

Example 4.4. Consider the differential equation (4.11) with $\gamma = \frac{1}{2}$, i.e., the sublinear ordinary differential equation

$$x^{(n)}(t) + a(t)\,|x(t)|^{1/2}\,\mathrm{sgn}x(t) = 0. \tag{4.13}$$

As it concerns the BVP (4.13), (1.2) and (1.3), condition (4.12) becomes

$$c - \left\{ \frac{1}{[(n-1)!]^{1/2}} \int_0^\infty t^{(n-1)/2}\,|a(t)|\,dt \right\} c^{1/2} - |\xi| \geq 0. \tag{4.14}$$

Let the following condition be satisfied:

$$\int_0^\infty t^{(n-1)/2}\,|a(t)|\,dt < \infty. \tag{4.15}$$

Following the lines of Example 1 in the author's paper [29], we can show that (4.14) is valid (as an equality) for

$$c = \left(\frac{1}{2\,[(n-1)!]^{1/2}} \int_0^\infty t^{(n-1)/2}\,|a(t)|\,dt + \sqrt{\left\{ \frac{1}{2\,[(n-1)!]^{1/2}} \int_0^\infty t^{(n-1)/2}\,|a(t)|\,dt \right\}^2 + |\xi|} \right)^2. \tag{4.16}$$

By taking into account (4.15), we can easily see that the formula (4.16) defines a real number c with $c > |\xi|$. So, we obtain the following result.

Assume that condition (4.15) *is satisfied, and let c be the real number given by* (4.16) (*note that $c > |\xi|$*). *Then the BVP* (4.13), (1.2) *and* (1.3) *has at least one solution x such that* (2.3) *holds.*

Now, we choose $n = 3$, $\xi = \frac{15}{4}$, and $a(t) = 2\sqrt{2}/(t+1)^3$ for $t \geq 0$. Then we can see that condition (4.15) is satisfied, and that (4.16) gives $c = \frac{25}{4}$. Hence, we arrive at the following result: *The boundary value problem*

$$\begin{cases} x'''(t) + \frac{2\sqrt{2}}{(t+1)^3}\,|x(t)|^{1/2}\,\mathrm{sgn}x(t) = 0 \\ x(0) = x'(0) = 0, \qquad \lim_{t\to\infty} x''(t) = \frac{15}{4} \end{cases}$$

has at least one solution x such that, for every $t \geq 0$,

$$\frac{5}{8}t^2 \leq x(t) \leq \frac{25}{8}t^2, \quad \frac{5}{4}t \leq x'(t) \leq \frac{25}{4}t, \quad \textit{and} \quad \frac{5}{4} \leq x''(t) \leq \frac{25}{4}.$$

Example 4.5. Let us consider the case of the differential equation (4.11) with $\gamma = 2$, i.e., the case of the superlinear ordinary differential equation

$$x^{(n)}(t) + a(t)[x(t)]^2 \operatorname{sgn} x(t) = 0. \tag{4.17}$$

In the case of the BVP (4.17), (1.2) and (1.3), condition (4.12) is written as

$$\left\{ \frac{1}{[(n-1)!]^2} \int_0^\infty t^{2(n-1)} |a(t)|\, dt \right\} c^2 - c + |\xi| \leq 0. \tag{4.18}$$

Let us suppose that

$$\int_0^\infty t^{2(n-1)} |a(t)|\, dt < \infty. \tag{4.19}$$

After a long analysis similar to that used by the author in Example 2 in [29], we can be led to the conclusion that, if

$$\frac{1}{[(n-1)!]^2} \int_0^\infty t^{2(n-1)} |a(t)|\, dt \leq \frac{1}{4|\xi|}, \tag{4.20}$$

then (4.18) holds (as an equality) for

$$c = \frac{1 - \sqrt{1 - 4\left\{\frac{1}{[(n-1)!]^2} \int_0^\infty t^{2(n-1)} |a(t)|\, dt\right\} |\xi|}}{\frac{2}{[(n-1)!]^2} \int_0^\infty t^{2(n-1)} |a(t)|\, dt}. \tag{4.21}$$

We notice that (4.20) implies condition (4.19). By the use of (4.20), it is not difficult to verify that the formula (4.21) defines a real number c with $c > |\xi|$. Thus, we get the next result.

Assume that condition (4.20) *is satisfied, and let c be the real number given by* (4.21) (*note that $c > |\xi|$*). *Then the BVP* (4.17), (1.2) *and* (1.3) *has at least one solution x such that* (2.3) *holds.*

Now, set $n = 3$, $\xi = \frac{3}{8}$, and $a(t) = 10/(t+1)^6$ for $t \geq 0$. Then we can verify that condition (4.20) is fulfilled (as a strict inequality), and that (4.21) becomes $c = \frac{1}{2}$. Thus, the next result is true: *The boundary value problem*

$$\begin{cases} x'''(t) + \frac{10}{(t+1)^6}[x(t)]^2 \operatorname{sgn} x(t) = 0 \\ x(0) = x'(0) = 0, \qquad \lim_{t\to\infty} x''(t) = \frac{3}{8} \end{cases}$$

has at least one solution x such that, for every $t \geq 0$,

$$\frac{1}{8}t^2 \leq x(t) \leq \frac{1}{4}t^2, \quad \frac{1}{4}t \leq x'(t) \leq \frac{1}{2}t, \quad \text{and} \quad \frac{1}{4} \leq x''(t) \leq \frac{1}{2}.$$

Before closing this section, we shall give a specific example, in which Theorem 2.2 (more specifically, the particular result stated in Remark 4.3) is applied.

Example 4.6. Consider the differential equation (4.2) with $n = 3$, and $a_0(t) = a_1(t) = a_2(t) = 2/\left[3(t+1)^4\right]$ for $t \geq 0$, i.e., the third order linear ordinary differential equation

$$x'''(t) + \frac{2}{3(t+1)^4}[x(t) + x'(t) + x''(t)] = 0. \tag{4.22}$$

Together with (4.22), we specify the initial condition

$$x(0) = x'(0) = 0. \tag{4.23}$$

Moreover, along with (4.22), we impose the condition

$$\lim_{t\to\infty} x''(t) = 0 \tag{4.24}$$

or the condition

$$\lim_{t\to\infty} x''(t) = 5. \tag{4.25}$$

It is not difficult to check that condition (4.10) is satisfied. Consequently, we are led to the following result: *Let c be any positive real number. Then the BVP* (4.22)$-$(4.24) *has exactly one solution x with*

$$|x''(t)| \le c \quad \textit{for all } t \ge 0;$$

this unique solution is necessarily the zero solution $x(t) = 0$ for $t \ge 0$. Furthermore, we can see that, when $\xi = 5$, equation (4.7) gives $c = 9$. Hence, we derive the next result: *The BVP* (4.22), (4.23) *and* (4.25) *has exactly one solution x with*

$$|x''(t)| \le 9 \quad \textit{for all } t \ge 0;$$

this unique solution x is such that, for every $t \ge 0$,

$$\frac{1}{2}t^2 \le x(t) \le \frac{9}{2}t^2, \quad t \le x'(t) \le 9t, \quad \textit{and} \quad 1 \le x''(t) \le 9.$$

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